On the occurrence of sliding bifurcations in piecewise smooth dynamical systems

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We study the occurrence of an interesting class of bifurcations in piecewise smooth dynamical systems. These bifurcations, termed *sliding* bifurcations, are shown to be the mechanism underlying the formation of periodic solutions evolving partly within the system discontinuity set. Numerical evidence of the existence of these sliding orbits and their bifurcations is presented. A possible framework to carry out their analytical investigation is also proposed.

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Piecewise smooth dynamical systems are increasingly used in many different branches of applied science to model the most diverse physical devices [1–5]. These systems are typically described by set of ODEs of the form

$$\dot{x} = f(x, t, p) \tag{1}$$

where $x \in \mathcal{R}^n$ is the state vector, $p \in \mathcal{R}^m$ is the parameter vector and $f : \mathcal{R}^{(n+m+1)} \mapsto \mathcal{R}^n$ is piecewise smooth.

From a geometric viewpoint, the phase space of (1) can be divided into countably many regions, say $G_i, i = 1, 2, .., k$. In each region, the system is described by a smooth functional form. At the boundaries, Φ_i between these regions, the trajectory of the system is continuous but non-differentiable, since the system switches to a different configuration whenever the system trajectory crosses one of these boundaries.

It has been shown that a family of nonstandard bifurcations, named border-collisions, often characterise the dynamical behaviour of a piecewise smooth system [6,7]. Specifically, border-collision bifurcations occur whenever the system trajectory (or part of it) becomes tangent to one of the phase space boundaries Φ_i. When this occurs, seemingly exotic dynamical transitions are often observed which are unique to nonsmooth dynamical systems (as for instance the sudden transition from a periodic solution to a chaotic evolution).

It is also well known that a piecewise smooth system can exhibit a peculiar type of solution, the so-called *sliding* motion [8,9]. This solution is characterised by lying within the system discontinuity set (boundaries in phase space) and can be heuristically seen as associated to an infinite number of switchings between different system configurations. Specifically, suppose that the direction of the system vector field points towards the switching hyperplane, S, on both sides of it. Then, when the system trajectory hits the switching surface, it will be constrained to evolve on it, until the direction of the vector field on one side or the other changes. Hence, by studying the gradient of the system vector field in a neighborhood of S we can identify regions $\bar{S} \subset S$ where sliding is possible, which we will term sliding regions. It has been shown, that the system dynamics within a sliding region can be studied by looking at an appropriate reduced order system. This can be obtained by applying Utkin's equivalent control method [9] or Filippov's convex method [8].

As recently shown by the authors, periodic solutions of the system can connect the sliding set with itself, giving rise to so-called sliding orbits [10,11]. (Some additional evidence of the existence of sliding orbits was also reported independently in the Russian literature [12] and in [15].) These orbits are characterised by lying partially within the system discontinuity set and have been found in several systems of relevance in applications as for instance power converters [10], relay feedback systems [11] and friction oscillators [15].

In this Letter, we propose that the formation of periodic solutions with sliding is due to a novel, more general class of bifurcations in piecewise smooth dynamical systems. These bifurcations describe the metamorphosis of a regular (non-sliding) periodic solution into a sliding orbit. The occurrence of these *sliding bifurcations*, is analysed through appropriately defined Poincaré maps and numerical evidence of their existence is presented. As a simple representative example, we will consider the case of a nonsmooth system often used in applications, the relay feedback system [13].

It is relevant to point out that the analysis presented here can be applied, without major modifications, to other relevant systems where sliding solutions play an important role in organising the system dynamics [10]. We anticipate that sliding bifurcations are also the reason for the onset of stick-slip oscillations in friction oscillators [5].

The relay feedback systems of interest are of the form:

$$\dot{x} = Ax + Bu \tag{2}$$
$$y = Cx \tag{3}$$
$$u = -\operatorname{sign} y, \tag{4}$$

where $\operatorname{sign} y = 1$ if $y > 0$, $\operatorname{sign} y = -1$ if $y < 0$, and $\operatorname{sign} y \in [-1, 1]$ if $y = 0$. In what follows, for the sake of clarity, we will restrict our attention to the case of a third-order system characterised by the matrices (in canonical form):

$$A = \begin{pmatrix} -(2\zeta\omega + \lambda) & 1 & 0 \\ -(2\zeta\omega + \omega^2) & 0 & 1 \\ -\lambda\omega^2 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} k \\ 2k\sigma\rho \\ k\rho^2 \end{pmatrix}, \tag{5}$$

and $C = (1\ 0\ 0)^T$ where $k > 0$, $\lambda > 0$. In particular, the parameters values are fixed to $k = 1, \sigma = -1, \lambda = 1, \zeta = 1$ and $\rho \in (-20, 10)$.

Notice that in this case the switching hyperplane S can be defined as $S := \{x \in \mathcal{R}^n : Cx = 0\}$, while as shown later, the sliding region can be derived to be the strip $\bar{S} = \{x \in S : CAx < CB\}$. Some typical periodic solutions of this system, including a sliding orbit, are depicted in Fig. 2.

To study the existence and stability of these solutions we can introduce a set of appropriate Poincaré maps. Specifically, say $\Pi : S/\bar{S} \mapsto \bar{S}$ the map from the switching plane (excluding the sliding region) back to itself and let $\Sigma : \bar{S} \mapsto \partial\bar{S}$ be the *sliding* map from the sliding region to its boundary ($\subset S$). Then, a non-sliding solution can be described by the corresponding fixed point, x^* of the map Π, while a sliding orbit by the fixed point of the composition $\Pi \circ \Sigma$. Hence, a set of necessary conditions of existence for periodic orbits, with or without sliding, can be obtained by looking for fixed points of these Poincaré maps. Moreover, their stability properties can be investigated by deriving the map Jacobian in a neighborhood of these fixed points.

For instance, in the case of the third-order relay system considered here, the Poincaré map, $\Pi : (t_n, x_n) \mapsto (t_{n+1}, x_{n+1})$, from the switching plane to itself can be easily constructed. (Note that is often convenient to embed the switching time instants, t_n, in the map definition.) Specifically, given a generic initial condition on the switching plane, say $x_0 = x(0) \in S$, the solution of (2)-(4) can be easily written as

$$x(t) = N(t)x_0 - M(t) \tag{6}$$

for $u = -1$, or

$$x(t) = N(t)x_0 + M(t) \tag{7}$$

for $u = +1$, where $N(t) = \exp(At)$ and $M(t) = \big(N(t) - I\big)A^{-1}B$, assuming that A is nonsingular. Therefore, starting from the pair (t_n, x_n), assuming $u = -1$, by using (6), the state at the successive switching can be written as

$$\tilde{x}_n = N(\delta_{n_1})x_n - M(\delta_{n_1}), \tag{8}$$

where δ_{n_1} is the time interval from t_n to the next switching time instant, i.e., $\tilde{x}_n = x(t_n + \delta_{n_1})$. The time variable δ_{n_1} is implicitly defined by the following switching condition:

$$C\tilde{x}_n = C\left[N(\delta_{n_1})x_n - M(\delta_{n_1})\right] = 0. \tag{9}$$

After $t = t_n + \delta_{n_1}$ the system will evolve on the other side of the switching plane, i.e., the output will become negative and the input will be $u = 1$. The state at the next switching time instant $x_{n+1} = x(t_{n+1})$ will then be given by

$$x_{n+1} = N(\delta_{n_2})\tilde{x}_n + M(\delta_{n_2}) \tag{10}$$

where $t_{n+1} = t_n + \delta_{n_1} + \delta_{n_2}$ and δ_{n_2} is implicitly defined by the following switching condition:

$$\begin{aligned} Cx_{n+1} = C\,[N(\delta_{n_2})N(\delta_{n_1})x_n - \\ N(\delta_{n_2})M(\delta_{n_1}) + M(\delta_{n_2})] = 0. \end{aligned} \tag{11}$$

The Poincaré map can be obtained by substituting (8) in (10), thus providing

$$x_{n+1} = N(\delta_{n_2})N(\delta_{n_1})x_n - N(\delta_{n_2})M(\delta_{n_1}) + M(\delta_{n_2}), \tag{12}$$

where δ_{n_1} and δ_{n_2} are implicitly defined by (9) and (11).

Similarly the *sliding map* Σ can be constructed by considering the equations of the reduced order system describing the system evolution within the sliding region. In the case of the relay feedback system considered here, the equations of such a reduced order system, obtained by applying Utkin's equivalent control method [9] are derived to be:

$$\dot{z} = \begin{pmatrix} -2\sigma\rho & 1 \\ -\rho^2 & 0 \end{pmatrix} z, \tag{13}$$

where the state z consists of the second and third components of x. The application of Utkin's method also allows the derivation of the sliding region. In facts, in order to have sliding, the method requires the system states to satisfy the following constraint:

$$\left|(CB)^{-1}CAx\right| < 1. \tag{14}$$

By substituting the system matrices (5) in (14), it is straightforward to see that (14) corresponds to $|x_2| < 1$. Hence, in this case the sliding region corresponds to a strip on S whose boundaries are defined by $|x_2| < 1$ (see fig. 2). As shown in Fig. 2(b) for some values of the system parameters a stable periodic trajectory connects the sliding strip with itself.

Using the Poincaré maps derived previously, we can now obtain conditions for the existence and local stability of the system periodic solutions. We will detail the derivation to the case of nonsliding solutions. Similar conditions for sliding orbits can be readily derived by appropriately modifying the expression for the non-sliding orbit given below.

It is simple to show that the equilibrium point of the Poincaré map (12) can be written as

$$\bar{x} = \left[I - N(2\bar{\delta})\right]^{-1} \left[I - N(\bar{\delta})\right] M(\bar{\delta}), \tag{15}$$

where we assumed that the orbit is symmetric, i.e., $\delta_{n_1} = \delta_{n_2} = \bar{\delta}$. A necessary condition for the existence of such a periodic solution is given by (9) at steady state, i.e., by the scalar equation

$$C\left[N(\bar{\delta})\left(I - N(2\bar{\delta})\right)^{-1}\left(I - N(\bar{\delta})\right) M(\bar{\delta}) - M(\bar{\delta})\right] = 0. \tag{16}$$

By solving (16) we obtain candidate time intervals $\bar{\delta}$ (and the corresponding fixed points from (15)) for possible limit cycles. Once a candidate $\bar{\delta}$ has been obtained, the existence of the corresponding orbits and their stability must be verified.

To investigate the stability of the periodic solutions located using the necessary conditions of existence outlined above, we now illustrate the derivation of the Jacobian of the Poincaré map around an equilibrium point corresponding to a given periodic solution.

Introducing the vector $\Delta_n = (\delta_{n_1}, \delta_{n_2})^T$, map (12) can be rewritten as follows:

$$x_{n+1} = f(x_n, \Delta_n) \tag{17}$$

and the switching conditions (9) and (11) can be rewritten in vector form as

$$\mu(x_n, \Delta_n) = 0. \tag{18}$$

By using implicit differentiation, the Jacobian can be computed as

$$J = \frac{df}{dx_n} = \frac{\partial f}{\partial x_n} - \frac{\partial f}{\partial \Delta_n}\left(\frac{\partial \mu}{\partial \Delta_n}\right)^{-1}\frac{\partial \mu}{\partial x_n}.$$

After some algebraic manipulation, we get

$$J = N_2 N_1 - \frac{1}{C\dot{\tilde{x}}_n^- C\dot{x}_{n+1}^-} N_1 \left[\dot{\tilde{x}}_n^- C\dot{x}_{n+1}^- C + \dot{\tilde{x}}_n^+ C\left(\dot{\tilde{x}}_n^- C N_2 - N_2 \dot{\tilde{x}}_n^- C\right)\right] N_1, \tag{19}$$

where $N_2 = N(\delta_{n_2})$, $N_1 = N(\delta_{n_1})$, $\dot{\tilde{x}}_n^- = A\tilde{x}_n - B$, $\dot{\tilde{x}}_n^+ = A\tilde{x}_n + B$, and $\dot{x}_{n+1}^- = Ax_{n+1} + B$. Note that the Jacobian is derived for the general case with asymmetric orbits. Eq. (19) can be used to check stability of symmetric orbits by assuming that $x_{n+1} = x_n = \bar{x}$ and $\delta_{n+1} = \delta_n = \bar{\delta}$ in (19) and then computing the eigenvalues of the corresponding Jacobian [11].

We, now, seek to uncover the mechanisms underlying the formation of sliding orbits by carrying out an investigation of the system parameter space. In so doing, we will restrict our attention to the case of the third order relay feedback system introduced above, described by eq. (2)-(4). Preliminary evidence for the existence of similar scenarios in other piecewise smooth dynamical systems has also been reported in [10].

Careful numerical computations show that as ρ is varied, the system undergoes several bifurcations. These are summarised in the bifurcation diagram depicted in Fig. 1, where the second component of the Poincaré map is shown versus ρ.

For decreasing values of ρ, we see that a non-sliding orbit turns into a sliding orbit at $\rho \approx 2.1$. Then, for $\rho \in (-9.4, 0)$ the origin is globally stable while for $\rho < -9.4$ a non-sliding orbit is present. The transition from a non-sliding to a sliding orbit can be better outlined when the phase-space evolution of the system is investigated on both side of the *sliding bifurcation point.* In particular, as shown in Figs. 2–3, we observed that the fixed point of the switching map corresponding to the non-sliding orbit enters the sliding strip at the bifurcation point and sliding orbits are then generated for further parameter variations.

Using the maps presented in the previous section, we can now locate the sliding bifurcation point analytically. In facts, at the bifurcation point the fixed point of the Poincaré map Π, say x^*, corresponding to the non-sliding orbit undergoing the bifurcation, lies on the boundary of the sliding region $\partial\bar{S}$. Thus, the exact bifurcation point ρ^* can be isolated by imposing this extra condition on the map, together with the necessary conditions of existence for non-sliding solutions (16). In the case of the relay feedback system considered previously this additional constraint is $x_2 = \pm 1$.

Solving this system of equations using a numerical package, such as Maple, the bifurcation point (i.e. the parameter value at which the fixed point sits on the boundary of the sliding strip) was found to be $\rho^* = 2.098841$.

For further parameter variation, the newly formed sliding orbit is characterised by a longer and longer sliding section until for $\rho = 0$ the origin becomes the only stable attractor. As depicted in Fig. 1, this remains the only equilibrium until at $\rho \approx -9.4$, a new simple limit cycle is generated after a saddle-node bifurcation. Again, using the analytical tools devised above, this bifurcation was located analytically and shown to be a saddle-node bifurcation of the system switching map, characterised by having both the eigenvalues of the map Jacobian crossing the unit circle at +1. This bifurcation, actually, generates a pair of equilibria, one stable and the other unstable, hence the existence of a corresponding stable and unstable simple orbits was also detected.

A further bifurcation at $\rho \approx 0.5$ was also detected and recently identified as a symmetry breaking bifurcation giving rise to a branch of highly unstable asymmetric solutions (for further details see [14]).

In conclusion, in this Letter, we have discussed the occurrence of a general class of bifurcations in piecewise smooth dynamical systems, describing the formation of so-called sliding orbits. In particular, the *sliding bifurcation* was defined as the transition from a simple limit cycle to a sliding orbit, occurring when the fixed point ,x^*, of the Poincaré map, Π, crosses the boundary of the sliding region on the switching hyperplane S, i.e. $x^* \in \partial \bar{S}$. A method to analyse the existence and stability of these orbits and to locate analytically the sliding bifurcation point has been presented.

As shown in [10], sliding orbits can be particularly relevant in affecting the dynamical behaviour of a piecewise smooth dynamical system. In facts, sliding orbits have been shown to organise bifurcation diagrams of periodic orbits characterised by an increasing number of switchings which have the shape of an intertwined double spiral. Thus, we believe that the further study of these bifurcation could be particularly relevant in applications. Current work is addressing the derivation of an appropriate local analysis around the bifurcation point (normal form derivation). Moreover, some preliminary evidence shows the occurrence of more complex phenomena due to sliding bifurcations such as the formation of multi-sliding orbits and seemingly unpredictable dynamics.

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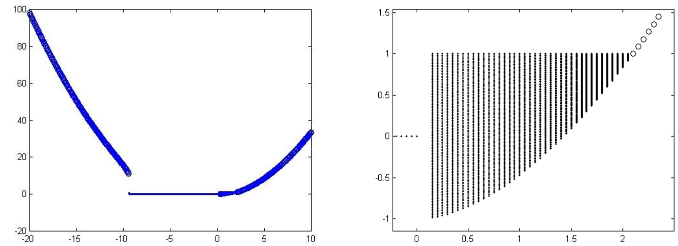
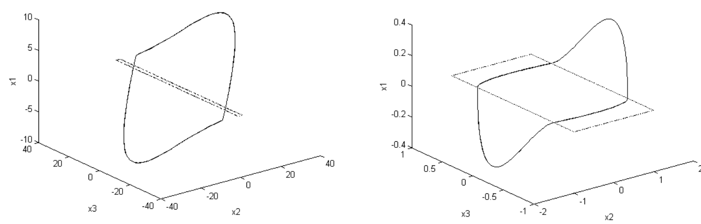


(a) (b)

FIG. 1. Bifurcation diagram of system (3), when ρ is varied. The second component of the Poincaré map is plotted against ρ in (a). A zoom of the transition from simple to sliding orbit is shown in (b).



(a) (b)

FIG. 2. Phase space diagrams before (a) and after (b) the bifurcation point with $\rho = 3$ (a) and $\rho = 1$ (b). The apparent change of the size of the sliding strip is only due to a change of scale.

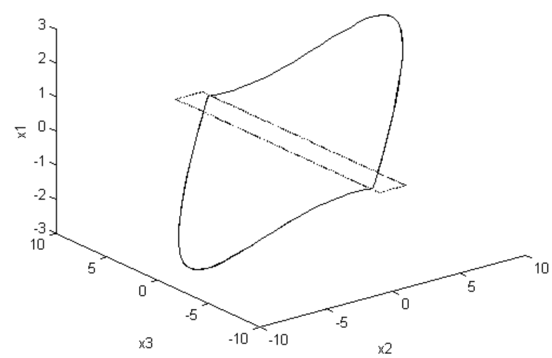


FIG. 3. Phase space diagrams of the simple limit cycle at the bifurcation point ($\rho \approx 2.1$). It can be clearly seen that the orbit intersects the switching plane on the boundary of the sliding strip